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On Hermite-Hadamard Type Inequalities for φ-convex Functions via Fractional Integrals

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ABSTRACT

In this paper, we establish integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals for φ-convex functions and some new inequalities of right-hand side of Hermite-Hadamard type are given for functions whose first derivatives absolute values φ-convex functions via Riemann-Liouville fractional integrals.

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1. INTRODUCTION

The function $f:[a,b]\subset\mathbf{R}\to\mathbf{R}$ is said to be convex if the following inequality holds:

$$f(\lambda x+(1-\lambda)y)\le\lambda f(x)+(1-\lambda)f(y)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. We say that f is concave if $(-f)$ is convex.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are very important in the literature (see Pecaric, 1992 and

Dragomir, 2000). These inequalities state that if $f : I \to \mathbf{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, Azpeitia, 1994, Dragomir, 1998; 2000 and the references cited therein.

In Dragomir, 1998, Dragomir and Agarwal proved the following results connected with the right part of (1).

Lemma 1. Let $f : I^\circ \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)dt. \tag{2}$$

Theorem 1. Let $f : I^\circ \subseteq \mathrm{R} \to \mathrm{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(|f'(a)| + |f'(b)|\right). \tag{3}$$

Meanwhile, Sarikaya et al., 2013 presented the following important integral identity including the first-order derivative of f to establish many interesting Hermite-Hadamard type inequalities for convexity functions via Riemann-Liouville fractional integrals of the order $\alpha > 0$.

Lemma 2. Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b.$ If $f' \in L[a,b],$ then the following equality for fractional integrals holds:

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right]$$
$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)\,dt. \tag{4}$$

It is remarkable that Sarikaya et al., 2013 first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 2. Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b].$ If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2} \tag{5}$$

with $\alpha > 0.$

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult (Gorenflo, 1997, Miller, 1993).

Definition 1. Let $f \in L_1[a,b].$ The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \ge 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt, \quad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x).$

For some recent results connected with fractional integral inequalities see (Belarbi and Dahmani, 2009, Dahmani (2010; (2010-1(1)); (2010-1(2)); (2010-2(3)), Sarikaya, 2012, Tunc, 2013).

In Youness, 1999, have defined the φ -convex function as follows:

Definition 2. Let $\varphi : [a,b] \subset \mathbb{R} \to [a,b].$ A function $f : [a,b] \to \mathbb{R}$ is said to be $\varphi -$ convex on $[a,b]$ if, for every $x, y \in [a,b]$ and $\lambda \in [0,1]$, the following inequality holds:

$$f\left(\lambda\varphi(x) + (1-\lambda)\varphi(y)\right) \leq \lambda f\left(\varphi(x)\right) + (1-\lambda) f\left(\varphi(y)\right).$$

Obviously, if $\varphi(x) = x$, then the classical convexity is obtained from the previous definition.

In Sarikaya et. al., 2014 gave the following important inequalities for φ -convex mappings :

Theorem 3. Let J be an interval $a,b \in J$ with $a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a φ -convex function on $I = [a,b],$ then we have

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \leq \frac{1}{\varphi(b)-\varphi(a)} \int_{\varphi(a)}^{\varphi(b)} f(\varphi(x))\, d\varphi(x) \leq \frac{f(\varphi(a)) + f(\varphi(b))}{2}. \tag{6}$$

In this paper, by using φ -convex mappings, we give Hermite-Hadamard's inequalities for Riemann-Liouville fractional integral and some other integral inequalities using the identity is obtained for fractional integrals.

2. MAIN RESULTS

Hermite-Hadamard's inequalities can be represented in fractional integral forms as follows:

Theorem 4. Let J be an interval $a,b \in J$ with $a<b$ and $\varphi : J \to \mathrm{R}$ a continuous increasing function. Let $f : I \subseteq \mathrm{R} \to \mathrm{R}$ be a φ -convex function on $I=[a,b]$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} & f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ & \leq \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a))\right] \\ & \leq \frac{f(\varphi(a))+f(\varphi(b))}{2} \end{aligned} \tag{7}$$

with $\alpha > 0$.

Proof. Since f is a φ -convex function on $[a,b]$, we have for $\varphi(x),\varphi(y) \in [\varphi(a),\varphi(b)]$ with $\lambda = \frac{1}{2}$

$$f\left(\frac{\varphi(x)+\varphi(y)}{2}\right) \leq \frac{f(\varphi(x))+f(\varphi(y))}{2} \tag{8}$$

i.e., with $\varphi(x)=t\varphi(a)+(1-t)\varphi(b)$, $\varphi(y)=(1-t)\varphi(a)+t\varphi(b)$,

$$\begin{aligned} & 2f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\ & \leq f(t\varphi(a)+(1-t)\varphi(b)) + f((1-t)\varphi(a)+t\varphi(b)). \end{aligned} \tag{9}$$

Multiplying both sides of (9) by $t^{\alpha-1}$, then integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{2}{\alpha} f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \\
&\le \int_0^1 t^{\alpha-1} f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt + \int_0^1 t^{\alpha-1} f\left((1-t)\varphi(a)+t\varphi(b)\right)dt \\
&= \int_{\varphi(b)}^{\varphi(a)} \left(\frac{\varphi(b)-\varphi(u)}{\varphi(b)-\varphi(a)}\right)^{\alpha-1} f(\varphi(u)) \frac{d\varphi(u)}{\varphi(a)-\varphi(b)} \\
&\quad + \int_{\varphi(a)}^{\varphi(b)} \left(\frac{\varphi(v)-\varphi(a)}{\varphi(b)-\varphi(a)}\right)^{\alpha-1} f(\varphi(v)) \frac{d\varphi(v)}{\varphi(b)-\varphi(a)} \\
&= \frac{\Gamma(\alpha)}{\left(\varphi(b)-\varphi(a)\right)^{\alpha}} \left[J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a)) \right]
\end{aligned}$$

i.e.

$$f\left(\frac{\varphi(a)+\varphi(b)}{2}\right) \le \frac{\Gamma(\alpha+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\alpha}} \left[J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a)) \right]$$

$$f\left(t\varphi(a)+(1-t)\varphi(b)\right) \le tf(\varphi(a)) + (1-t) f(\varphi(b))$$

and

$$f\left((1-t)\varphi(a)+t\varphi(b)\right) \le (1-t) f(\varphi(a)) + tf(\varphi(b)).$$

By adding these inequalities we have

$$\begin{aligned}
&f\left(t\varphi(a)+(1-t)\varphi(b)\right) + f\left((1-t)\varphi(a)+t\varphi(b)\right) \\
&\le tf(\varphi(a)) + (1-t) f(\varphi(b)) + (1-t) f(\varphi(a)) + tf(\varphi(b)).
\end{aligned} \tag{10}$$

Then multiplying both sides of (10) by $t^{\alpha-1}$ and integrating the resulting inequality with respest to t over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 t^{\alpha-1} f\left(t\varphi(a)+(1-t)\varphi(b)\right)dt + \int_0^1 t^{\alpha-1} f\left((1-t)\varphi(a)+t\varphi(b)\right)dt \\
&\le \left[f(\varphi(a)) + f(\varphi(b)) \right] \int_0^1 t^{\alpha-1} dt
\end{aligned}$$

i.e.

$$\frac{\Gamma(\alpha)}{\left(\varphi(b)-\varphi(a)\right)^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a))\right]\leq\frac{f\left(\varphi(a)\right)+f\left(\varphi(b)\right)}{\alpha}.$$

The proof is completed.

Remark. If in Theorem 4, we let $\alpha=1$, then the inequalities (7) become the inequalities (6) of Theorem 3.

To prove our main results, we need the following lemma:

Lemma 3. Let J be an interval $a,b\in J$ with $0\leq a<b$ and $\varphi : J\to \mathrm{R}$ a continuous increasing function. Let $f : I\subset \mathrm{R}\to \mathrm{R}$ be a differantiable function on I° (the interior I).If $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$, then the following equality holds:

$$\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a))\right] \tag{11}$$

$$=\frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(t\varphi(a)+(1-t)\varphi(b)\right)dt.$$

where $\alpha>0$.

Proof. It suffices to note that

$$\begin{aligned} S &= \int_0^1\left[(1-t)^{\alpha}-t^{\alpha}\right]f'\left(t\varphi(a)+(1-t)\varphi(b)\right)dt \\ &= \left[\int_0^1(1-t)^{\alpha}f'\left(t\varphi(a)+(1-t)\varphi(b)\right)dt\right] \\ &\quad +\left[-\int_0^1 t^{\alpha}f'\left(t\varphi(a)+(1-t)\varphi(b)\right)dt\right] \qquad (12) \\ &= S_1+S_2. \end{aligned}$$

Integrating by parts

$$S_1 = \int_0^1 (1-t)^\alpha f'(t\varphi(a)+(1-t)\varphi(b))dt$$
$$= \int_0^1 (1-t)^\alpha f'(t\varphi(a)+(1-t)\varphi(b))dt$$

$$= (1-t)^\alpha \left. \frac{f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)} \right|_0^1$$

$$+\int_0^1 \alpha(1-t)^\alpha \frac{f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)} dt \tag{13}$$

$$= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)} \int_{\varphi(b)}^{\varphi(a)} \left(\frac{\varphi(a)-\varphi(x)}{\varphi(a)-\varphi(b)} \right)^{\alpha-1} \frac{f(\varphi(x))}{\varphi(a)-\varphi(b)} d\varphi(x)$$

$$= \frac{f(\varphi(b))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^\alpha_{\varphi(b)^-} f(\varphi(a))$$

and similarly we get,

$$S_2 = -\int_0^1 t^\alpha f'(t\varphi(a)+(1-t)\varphi(b))dt$$

$$= -\left. \frac{t^\alpha f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)} \right|_0^1 + \alpha \int_0^1 t^{\alpha-1} \frac{f(t\varphi(a)+(1-t)\varphi(b))}{\varphi(a)-\varphi(b)} dt$$

$$= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\alpha}{\varphi(b)-\varphi(a)} \int_{\varphi(b)}^{\varphi(a)} \left(\frac{\varphi(b)-\varphi(x)}{\varphi(b)-\varphi(a)} \right)^{\alpha-1} \frac{f(\varphi(x))}{\varphi(a)-\varphi(b)} d\varphi(x) \tag{14}$$

$$= \frac{f(\varphi(a))}{\varphi(b)-\varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}} J^\alpha_{\varphi(a)^+} f(\varphi(b)).$$

Using (13) and (14) in (12), it follows that

$$S = \frac{f(\varphi(a)) + f(\varphi(b))}{\varphi(b) - \varphi(a)} - \frac{\Gamma(\alpha+1)}{(\varphi(b)-\varphi(a))^{\alpha+1}}\left[J^{\alpha}_{\varphi(a)^+} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^-} f(\varphi(a))\right]$$

Thus, by multiplying the both sides by $\frac{\varphi(b)-\varphi(a)}{2}$, we have the conclusion (11).

Remark. If we take $\varphi(x) = x$ in Lemma 3, then the inedentity (11) reduces to the identity (4).

By using this Lemma, we can obtain the following fractional integral inequality:

Theorem 5. Let J be an interval $a,b \in J$ with $0 \le a < b$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differantiable function on I° (the interior I) and $f' \in L_1[\varphi(a), \varphi(b)]$ for $\varphi(a), \varphi(b) \in I$. If $|f'|^q$ is the φ - convex on $[a,b]$, $q \ge 1$, then the following inequaliy holds:

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))^+} f(\varphi(b)) + J^{\alpha}_{(\varphi(b))^-} f(\varphi(a)\right]\right|$$

$$\le \frac{\varphi(b)-\varphi(a)}{2^{\frac{2-q}{q}}}\left[\frac{1}{\alpha+1}\left(1 - \frac{1}{2^{\alpha}}\right)\right]\left[|f'(\varphi(a))|^q + |f'(\varphi(b))|^q\right]^{\frac{1}{q}}. \tag{15}$$

where $\alpha > 0$.

Proof. Firstly, we suppose that $q = 1$. Using Lemma 3 and φ -convexity of $|f'|^q$, we find that

$$\left|\frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))^+} f(\varphi(b)) + J^{\alpha}_{(\varphi(b))^-} f(\varphi(a))\right]\right|$$

$$\le \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^{\alpha} - t^{\alpha}\right|\left|f'\left[t(\varphi(a)) + (1-t)(\varphi(b))\right]\right| dt \tag{16}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2}\int_0^1 \left|(1-t)^\alpha - t^\alpha\right|\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right]dt$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\left\{\int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right]dt\right.$$

$$\left.+\int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right]dt\right\}$$

$$= \frac{\varphi(b)-\varphi(a)}{2}\left\{K_1 + K_2\right\}.$$

Hence, conculating K_1 ve K_2, we have

$$K_1 = \int_0^{\frac{1}{2}} \left[(1-t)^\alpha - t^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right]dt$$

$$= \left|f'(\varphi(a))\right|\int_0^{\frac{1}{2}} \left[(1-t)^\alpha t - t^{\alpha+1}\right]dt + \left|f'(\varphi(b))\right|$$

$$\int_0^{\frac{1}{2}} \left[(1-t)^{\alpha+1} - t^\alpha (1-t)\right]dt \tag{17}$$

$$= \left|f'(\varphi(a))\right|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] + \left|f'(\varphi(b))\right|$$

$$\left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right]$$

and

$$K_2 = \int_{\frac{1}{2}}^1 \left[t^\alpha - (1-t)^\alpha\right]\left[t\left|f'(\varphi(a))\right| + (1-t)\left|f'(\varphi(b))\right|\right]dt$$

$$= \left|f'(\varphi(a))\right|\left[\frac{1}{\alpha+2} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right] + \left|f'(\varphi(b))\right| \tag{18}$$

$$\left[\frac{1}{(\alpha+2)(\alpha+1)} - \frac{1}{2^{\alpha+1}(\alpha+1)}\right].$$

Using (17) and (18) in (16), it follows that

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))^{+}}f(\varphi(b))+J^{\alpha}_{(\varphi(b))^{-}}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\frac{1}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\left[|f'(\varphi(a))|+|f'(\varphi(b))|\right]$$

Secondly, we suppose that $q>1$. Using Lemma 3 and power mean inequality, we obtain

$$\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|dt$$
$$\leq\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|^{q}dt\right)^{\frac{1}{q}}. \tag{19}$$

Hence, using φ -convexity of $|f'|^{q}$ and (19) we obtain

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{(\varphi(a))^{+}}f(\varphi(b))+J^{\alpha}_{(\varphi(b))^{-}}f(\varphi(a))\right]\right|$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|^{q}dt\right)^{\frac{1}{q}}$$
$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]dt+\int_{\frac{1}{2}}^{1}\left[t^{\alpha}-(1-t)^{\alpha}\right]dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left[t\left|f'(\varphi(a))\right|^{q}+(1-t)\left|f'(\varphi(b))\right|^{q}\right]dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2} 2^{\frac{q-1}{q}} \left[\frac{1}{\alpha+1}\left(1-\frac{1}{2^{\alpha}}\right)\right]\left[\left| f'(\varphi(a))\right|^{q} + \left| f'(\varphi(b))\right|^{q} \right]^{\frac{1}{q}}$$

which completes the proof.

Theorem 6. Let J be an interval $a,b \in J$ with $0 \leq a < b$ and $\varphi : J \to \mathrm{R}$ a continuous increasing function. Let $f : I \subset \mathrm{R} \to \mathrm{R}$ be a differantiable function on I° (the interior I) and $f' \in L_1\left[\varphi(a), \varphi(b)\right]$ for $\varphi(a), \varphi(b) \in I$. If $\left| f'\right|^{q}$ is the φ - convex on $[a,b]$, $q>1$, then the following inequaliy holds:

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\alpha}} \left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a)) \right] \right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2} \left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}} \left(\frac{\left| f'(\varphi(a))\right|^{q} + \left| f'(\varphi(b))\right|^{q}}{2} \right)^{\frac{1}{q}}. \qquad (20)$$

where $\alpha > 0$ and $\frac{1}{p} + \frac{1}{q} = 1.$

Proof. Using Lemma 3, φ -convexity of $\left| f\right|^{q}$ and well-known Hölder's inequality, we obtain

$$\left| \frac{f(\varphi(a)) + f(\varphi(b))}{2} - \frac{\Gamma(\alpha+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\alpha}} \left[J^{\alpha}_{\varphi(a)^{+}} f(\varphi(b)) + J^{\alpha}_{\varphi(b)^{-}} f(\varphi(a)) \right] \right|$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2} \int_{0}^{1} \left| (1-t)^{\alpha} - t^{\alpha} \right| \left| f'\left[t(\varphi(a)) + (1-t)(\varphi(b)) \right] \right| dt$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2} \left(\int_{0}^{1} \left| (1-t)^{\alpha} - t^{\alpha} \right|^{p} dt \right)^{\frac{1}{p}} \left(\int_{0}^{1} \left| f'\left[t(\varphi(a)) + (1-t)(\varphi(b)) \right] \right|^{q} dt \right)^{\frac{1}{q}}$$

$$\leq \frac{\varphi(b)-\varphi(a)}{2} \left(\int_{0}^{\frac{1}{2}} \left[(1-t)^{\alpha} - t^{\alpha} \right]^{p} dt + \int_{\frac{1}{2}}^{1} \left[t^{\alpha} - (1-t)^{\alpha} \right]^{p} dt \right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\left[t\left|f'(\varphi(a))\right|^q+(1-t)\left|f'(\varphi(b))\right|^q\right]dt\right)^{\frac{1}{q}}$$

$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha p}-t^{\alpha p}\right]dt+\int_{\frac{1}{2}}^1\left[t^{\alpha p}-(1-t)^{\alpha p}\right]dt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{\left|f'(\varphi(a))\right|^q+\left|f'(\varphi(b))\right|^q}{2}\right)^{\frac{1}{q}}$$

$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left[\frac{2}{\alpha p+1}\left(1-\frac{1}{2^{\alpha p}}\right)\right]^{\frac{1}{p}}\left(\frac{\left|f'(\varphi(a))\right|^q+\left|f'(\varphi(b))\right|^q}{2}\right)^{\frac{1}{q}}.$$

Here, we use $(A-B)^p\leq A^p-B^p$ for any $A>B\geq 0$ and $p\geq 1$.

Theorem 7. Let J be an interval $a,b\in J$ with $0\leq a<b$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differantiable function on I° (the interior I) and $f'\in L_1[\varphi(a),\varphi(b)]$ for $\varphi(a),\varphi(b)\in I$. If $|f'|^q$ φ - convex on $[a,b]$ for same fixed $q\geq 1$, then the following inequality for fractional integrals holds:

$$\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2(\varphi(b)-\varphi(a))^{\alpha}}\left[J^{\alpha}_{\varphi(a)^+}f(\varphi(b))+J^{\alpha}_{\varphi(b)^-}f(\varphi(a))\right]$$

$$\leq\left[\frac{1}{q\alpha+1}\left(1-\frac{1}{2^{q\alpha+1}}\right)\right]^{\frac{1}{q}}\left(\frac{\left|f'(\varphi(a))\right|^q+\left|f'(\varphi(b))\right|^q}{2}\right)^{\frac{1}{q}}$$

where $\alpha>0$.

Proof. Using Lemma 3, φ -convexity of $|f'|^q$, and well-known Hölder's inequality, we have

$$\left|\frac{f(\varphi(a))+f(\varphi(b))}{2}-\frac{\Gamma(\alpha+1)}{2\left(\varphi(b)-\varphi(a)\right)^{\alpha}}\left[J^{\alpha}_{\varphi(a)^{+}}f(\varphi(b))+J^{\alpha}_{\varphi(b)^{-}}f(\varphi(b))\right]\right|$$

$$\leq\frac{\varphi(b)-\varphi(a)}{2}\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|dt$$

$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^1 1^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|(1-t)^{\alpha}-t^{\alpha}\right|^q\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]^q\right|dt\right)^{\frac{1}{q}}$$

$$=\frac{\varphi(b)-\varphi(a)}{2}\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\alpha}-t^{\alpha}\right]^q\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b)\right]\right|^q dt\right.$$

$$\left.+\int_0^{\frac{1}{2}}\left[t^{\alpha}-(1-t)^{\alpha}\right]^q\left|f'\left[t(\varphi(a))+(1-t)(\varphi(b))\right]\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{\varphi(b)-\varphi(a)}{2}\left(\left|f'(\varphi(a))\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha}t-t^{q\alpha+1}\right]dt\right.$$

$$+\left|f'(\varphi(b))\right|^q\int_0^{\frac{1}{2}}\left[(1-t)^{q\alpha+1}-t^{q\alpha}(1-t)\right]dt$$

$$+\left|f'(\varphi(a))\right|^q\int_{\frac{1}{2}}^{1}\left[t^{q\alpha+1}-(1-t)^{q\alpha}t\right]dt$$

$$\left.+\left|f'(\varphi(b))\right|^q\int_{\frac{1}{2}}^{1}\left[t^{q\alpha}(1-t)-(1-t)^{q\alpha+1}\right]dt\right)^{\frac{1}{q}}$$

$$=\frac{\varphi(b)-\varphi(a)}{2}\left(\frac{1}{\alpha+1}\left[1-\frac{1}{2^{\alpha}}\right]\right)^{\frac{1}{q}}\left[\left|f'(\varphi(a))\right|+\left|f'(\varphi(b))\right|\right]^{\frac{1}{q}}.$$

Here, we use $(A-B)^p\leq A^p-B^p$, for any $A>B\geq 0$ and $q\geq 1$.

3. CONCLUSION

In this paper, we have presented a Hermite-Hadamard's inequalities for φ-convex functions via fractional integrals. We defined that lemma and we established new theorems with through this lemma.

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